Consider the parametric equations $\begin{aligned} x &= -2\ln t \\ y &= t^{-6} \end{aligned}$, $t \in [1, \infty)$.

SCORE: _____ / 30 PTS

[a] Eliminate the parameter. **Write your final answer in the form y as a simplified function of x.**

$-\frac{1}{2}x = \ln t$

$t = e^{-\frac{1}{2}x}$ ⑤

$y = (e^{-\frac{1}{2}x})^{-6}$ ⑤

$y = e^{3x}$ ⑤

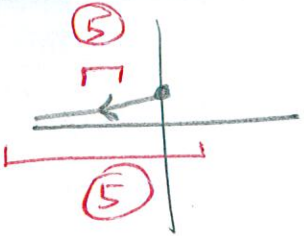
[b] Sketch the general shape and position of the graph of the resulting rectangular equation in part [a]. (Don't worry about specific $x-$ or $y-$ coordinates.)

⑤

[c] Highlight the part of the graph in part [b] which corresponds to the original parametric equations. Indicate clearly the orientation/direction of the resulting parametric curve.

AS t GOES FROM 1 TO ∞

X=-2ln t GOES FROM 0 TO -∞



⑤

⑤

Find the logarithmic formula for $\tanh^{-1} x$ by solving $x = \tanh y$ for y using the exponential definition and an algebraic substitution $z = e^{y}$.

SCORE: _____ / 20 PTS

$$x = \frac{e^{y} - e^{-y}}{e^{y} + e^{-y}} = \frac{z - \frac{1}{z}}{z + \frac{1}{z}} \cdot \frac{z}{z} = \frac{z^2 - 1}{z^2 + 1}$$

$$x z^2 + x = z^2 - 1$$

$$x + 1 = z^2 - x z^2 = (1 - x) z^2$$

$$z^2 = \frac{1 + x}{1 - x}$$

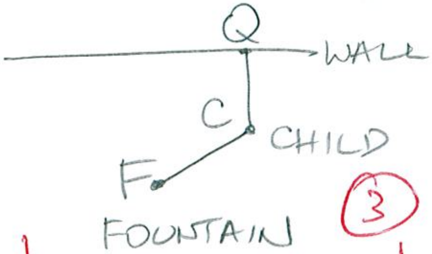
$$e^{2y} = \frac{1 + x}{1 - x}$$

$$2y = \ln \frac{1 + x}{1 - x} \longrightarrow y = \frac{1}{2} \ln \frac{1 + x}{1 - x} = \tanh^{-1} x$$

ALL ITEMS $2\frac{1}{2}$ POINTS

A drinking fountain is 6 feet from the wall of a school building. A child is running on the school grounds, so that it is always half as far from the fountain as it is from the wall. What is the shape of the child's path ? **SCORE: _____ / 10 PTS**

Draw a diagram and write algebraic equations involving distances to justify your answer.



③

$CF = \frac{1}{2} CQ$

$\frac{CF}{CQ} = \frac{1}{2} = e \longrightarrow$ ELLIPSE

④

③

AJ and BJ were working on their polar graphing partner quiz. **SCORE: _____ / 40 PTS**

On the question about the polar equation $r = 2 + 4\sin 3\theta$, they determined correctly that the symmetry tests $(-r, \pi - \theta)$, $(-r, \theta)$, $(-r, -\theta)$ and $(r, -\theta)$ do **NOT** indicate that the graph is symmetric.

AXIS POLE $\theta = \frac{\pi}{2}$ AXIS

[a] **Using their results, along with the tests and shortcuts shown in lecture**, test if the graph is symmetric over the pole, the polar axis and/or $\theta = \frac{\pi}{2}$. State your conclusions in the table. **NOTE: Run as FEW tests as needed to prove your answers are correct.**

POLE: $r = 2 + 4\sin 3(\pi + \theta)$ ③

③ $r = 2 + 4\sin(3\pi + 3\theta)$

$r = 2 + 4(\cancel{\sin 3\pi}^{\,0}\cos 3\theta + \cancel{\cos 3\pi}^{\,-1}\sin 3\theta)$

③ $r = 2 - 4\sin 3\theta$ ✗

$\theta = \frac{\pi}{2}$: $r = 2 + 4\sin 3(\pi - \theta)$ ③

② $r = 2 + 4\sin 3\theta$ ✓

↑ SIGN CHANGE FROM PRIOR TEST

Type of symmetry	Conclusion
Over the pole	② NO CONCLUSION
Over the polar axis	② NO CONCLUSION
Over $\theta = \frac{\pi}{2}$	② SYMMETRIC

[b] Based on the results of part [a], what is the minimum interval of the graph you need to plot (before using reflections to draw the rest of the graph) ?

$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ ⑤

[c] Find all angles **algebraically** in the minimum interval in part [b] at which the graph goes through the pole.

$r = 2 + 4\sin 3\theta = 0$

③ $\sin 3\theta = -\frac{1}{2}$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$-\frac{3\pi}{2} \le 3\theta \le \frac{3\pi}{2}$

⑨ $3\theta = -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6}$

③ $\theta = -\frac{5\pi}{18}, -\frac{\pi}{18}, \frac{7\pi}{18}$

Name the shape of the graphs of the following polar equations.
If the graph is a rose curve, state the number of petals.

SCORE: _____ **/ 20 PTS**

$r = \frac{\frac{10}{3}}{1-\frac{2}{3}\cos\theta}$

[a] $r = \dfrac{10}{3-2\cos\theta}$ ELLIPSE ③

$|2| = 2|-1|$

[b] $r = 2-\cos\theta$ CONVEX LIMAÇON ③

[c] $r = 9\sin 12\theta$ ROSE CURVE 24 PETALS ⑤

[d] $r = \frac{\pi}{2}$ CIRCLE ③

$|4| < |9|$

[e] $r = 4+9\sin\theta$ LIMAÇON WITH LOOP ③

$r = \frac{5}{1+\frac{3}{2}\sin\theta}$

[f] $r = \dfrac{10}{2+3\sin\theta}$ HYPERBOLA ③

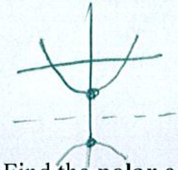
Rewrite $\operatorname{sech}\left(-\frac{1}{2}\ln x\right)$ in terms of exponential functions and simplify. **SCORE: _____ / 10 PTS**

$$\boxed{\frac{2}{e^{-\frac{1}{2}\ln x}+e^{\frac{1}{2}\ln x}}} = \boxed{\frac{2}{x^{-\frac{1}{2}}+x^{\frac{1}{2}}}} \cdot \frac{x^{\frac{1}{2}}}{x^{\frac{1}{2}}} = \boxed{\frac{2\sqrt{x}}{1+x}}$$

③ ④ ③

A hyperbola has a focus at the pole and vertices with **rectangular** co-ordinates $(0, -3)$ and $(0, -15)$. SCORE: _____ / 20 PTS

[a] Find polar co-ordinates for the vertices, using positive values of r and θ.



$(3, \frac{3\pi}{2})$ $(15, \frac{3\pi}{2})$ ②

↓ USE $(-15, \frac{\pi}{2})$

[b] Find the **polar** equation of the hyperbola.

$$r = \frac{ep}{1 - e\sin\theta}$$ ④

$$3 = \frac{ep}{1+e}$$ ②

$$-15 = \frac{ep}{1-e}$$ ②

$ep = 3+3e$ $ep = -15+15e$

$3+3e = -15+15e$ ②

$e = \frac{3}{2}$ ②

$\frac{3}{2}p = 3 + \frac{9}{2} \longrightarrow p = 5$ ②

② $$r = \frac{\frac{3}{2}\cdot 5}{1-\frac{3}{2}\sin\theta}\cdot\frac{2}{2}$$

② $$r = \frac{15}{2-3\sin\theta}$$